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Theory of the almost-highest wave: the inner solution

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This paper investigates the flow near the summit of steep, progressive gravity wave when the crest is still rounded but the flow is approaching Stokes's corner flow. The natural length scale in the neighbourhood of the summit is seen to be $l = q^2/2g$, where g denotes gravity and q is the particle speed at the crest in a reference frame moving with the wave speed. We show that a class of self-similar smooth local flows exists which satisfy the free-surface condition and which tend to Stokes's corner flow when the radial distance r becomes large compared with l. The behaviour of the solution at large values of r/l is shown to depend on the roots of the transcendental equation

$$K \tanh K = \pi/2\sqrt{3}.$$

The two real roots correspond to a damped oscillation of the free surface decaying like $(l/r)^{\frac{1}{2}}$. The positive imaginary roots correspond to perturbations vanishing like higher negative powers of r.

The complete flow is calculated by transforming the domain onto the interior of a circle in the complex plane and expanding the potential at the surface in a Fourier series. The computation is checked by an independent method, based on approximating the flow by a sequence of dipoles. The profile of the surface is found to intersect its asymptote at large values of r/l. This implies that the maximum slope slightly exceeds 30°. The computed value 30·37° is in close agreement with that obtained by extrapolating the maximum slopes of steep gravity waves, as calculated by previous authors. The vertical acceleration of a particle at the crest is $0{\cdot}388g$. In the far field, however, the acceleration tends to the value $\frac{1}{2}g$ corresponding to the Stokes corner flow.

1. Introduction

Though the theory of water waves of low or moderate steepness is in many respects well developed, the situation is quite otherwise for surface waves whose steepness is such that the waves are close to breaking. Even for steady progressive irrotational waves, when surface tension and viscosity are both neglected, the problem is made both difficult and interesting by the nonlinearity of the condition that the pressure must be a constant at the free surface. A possible limiting form for the crest of a gravity wave in which the free surface forms a *sharp* corner with a 120° internal angle was suggested by Stokes (1880). This local solution has been used as the starting-point for calculations of the complete form of the steepest progressive wave in deep water by

Michell (1893), Yamada (1957*a*), Schwartz (1974) and others; and for the steepest solitary wave by Yamada (1957*b*), Lenau (1966) and Schwitters (1966). Some simple approximations to the limiting wave were given by Longuet-Higgins (1973, 1974).

These calculations, however, refer only to the steepest possible waves. What is the form of waves that are steep but do not yet have a sharp angle at the crest? Here the small-amplitude expansions of Stokes for periodic waves and Rayleigh for solitary waves, though they yield surprising and interesting results (Schwartz 1974; Longuet-Higgins & Fenton 1974; Longuet-Higgins 1975; Cokelet 1976) are mathematically very inconvenient. The same is also true of the numerical techniques used by Sasaki & Murakami (1973) and the integral-equation method of Byatt-Smith & Longuet-Higgins (1976), both of which involve computations of increasing length as the limit of a sharp-crested wave is approached.

An attempt to calculate the form of waves having *nearly* the limiting amplitude was first made by Havelock (1918) by perturbing Michell's solution for the highest wave. But Grant (1973) has pointed out that the analytical structure of the highest wave must be more complicated than was assumed by Havelock.

In this paper we pose the following question. As a progressive gravity wave, of constant length, approaches its maximum height, and *while the crest is still rounded*, does the flow near the wave crest have asymptotically some limiting form? In other words, if κ denotes the curvature at the crest and r the radial distance, is there a smooth flow with length scale of order κ^{-1}, having no sharp discontinuity in surface slope, which as κr tends to infinity approaches the Stokes 120° corner flow? Further, if such a flow exists is it unique?

Consider first the natural length scale for such a flow. Let the wave be reduced to a steady flow by reference to a frame moving with the phase velocity c. In this frame let q denote the speed of flow at the crest. For a sharp corner flow, q will vanish. Generally, when $q \neq 0$, an appropriate scale l for the local flow should be given by

$$l = q^2/2g. \tag{1.1}$$

In figure 1 we have taken the profiles of three different steep solitary waves, calculated by Byatt-Smith & Longuet-Higgins (1976), at equally spaced values of the parameter

$$\omega = 1 - q^2/gh, \tag{1.2}$$

where h denotes the undisturbed depth of water, and have rescaled them by using as the unit of length

$$l = q^2/2g = \tfrac{1}{2}(1-\omega)h. \tag{1.3}$$

It will be seen that the different profiles now lie close to each other and appear to approach a limiting curve, shown by the broken line.

Encouraged by this numerical evidence we proceed in §2 to a precise definition of the problem, and subsequently to a numerical solution, by two quite independent methods. In the first of these methods (§5) the velocity potential is approximated by a sequence of singularities (poles) situated above the free surface, whose strengths are adjusted so as to satisfy the constant-pressure condition at regularly spaced points along the free surface. For a good approximation, six dipoles are quite sufficient. In the second method (§6) the domain of the flow is transformed conformally onto the interior of a circle, and the space co-ordinate on the circumference is expanded in a

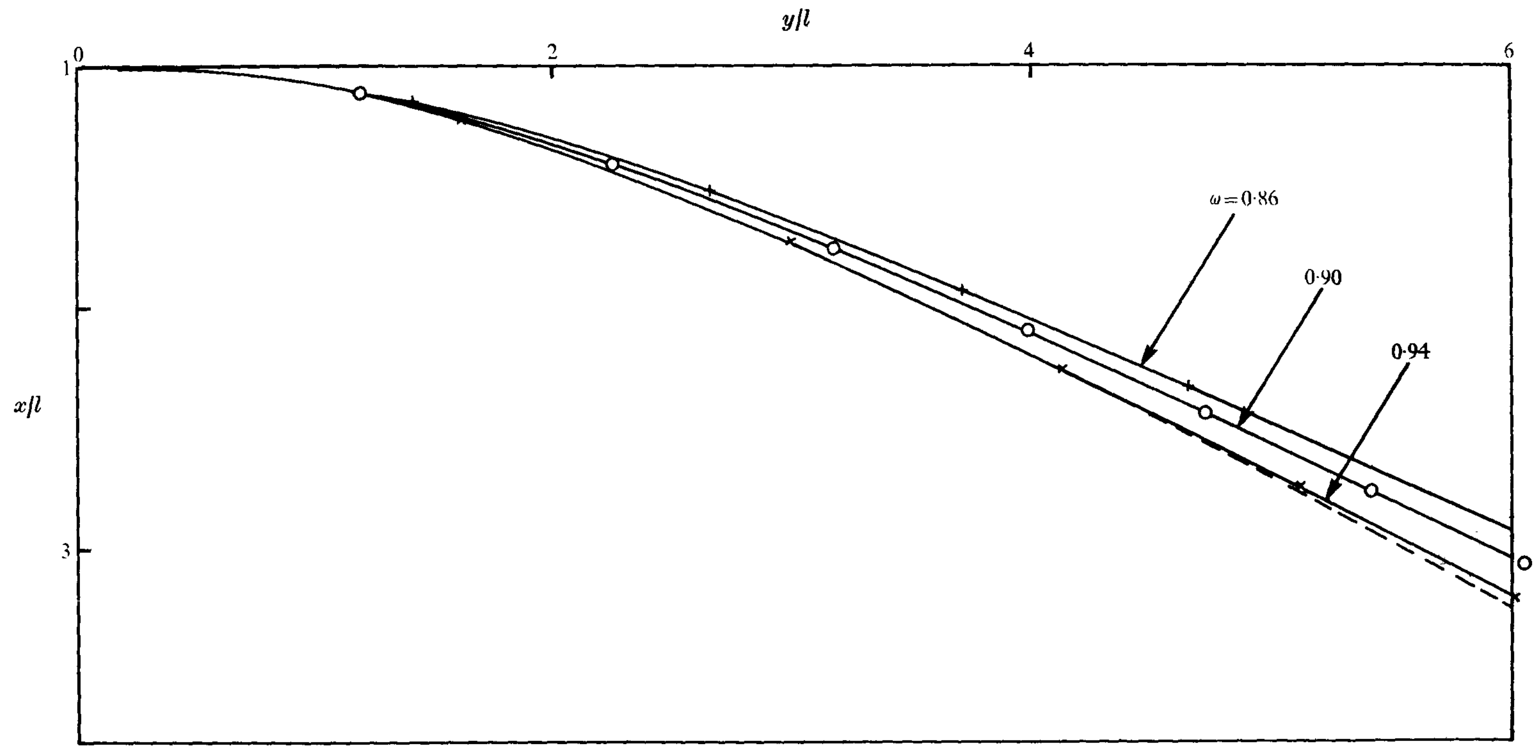

FIGURE 1. Profiles of the crests of steep solitary waves, scaled by the parameter $l = q^2/2g$ where q is the particle speed at the crest in a frame moving with the wave. The broken line corresponds to the solution of §§5 and 6.

Fourier series. The free-surface condition then gives an infinite sequence of nonlinear (cubic) equations to be satisfied by the coefficients. When solved numerically by truncation and successive approximation the solution rapidly converges. Moreover, we find very close agreement between this and the previous method, which strongly suggests that the solution to the problem is unique.

For large values of the dimensionless radius r/l the solution tends to the Stokes corner flow, not monotonically as was at first expected, but in an oscillatory manner (see figure 9). The period of oscillation is given by the real root of a simple transcendental equation (4.12). This implies that the maximum slope of the free surface very slightly exceeds that in the Stokes corner flow. The maximum angle is found to be 30·37°, which is checked with remarkable accuracy by an extrapolation of the recent results of Sasaki & Murakami (1973) both for solitary waves and for periodic waves in deep water. This conclusion has implications for certain existence proofs which have assumed the maximum slope angle not to exceed 30°.

2. Definition of the problem

In a frame of reference moving with the wave speed, take polar co-ordinates r, θ as in figure 2, with the origin O above the wave crest and the radius $\theta = 0$ directed vertically downwards. Writing

$$z = r\,e^{i\theta}, \quad \chi = \phi + i\psi, \tag{2.1}$$

where ϕ and ψ are the velocity potential and stream function, the pressure p is given by Bernoulli's equation

$$p + \tfrac{1}{2}|d\chi/dz|^2 - gr\cos\theta = C \tag{2.2}$$

(the density being taken as unity). On the free surface $\psi = 0$ the pressure is a constant, say zero. By vertical adjustment of the origin, the constant C may be made to vanish. Hence

$$|d\chi/dz|^2 = 2gr\cos\theta \quad \text{on} \quad \psi = 0. \tag{2.3}$$

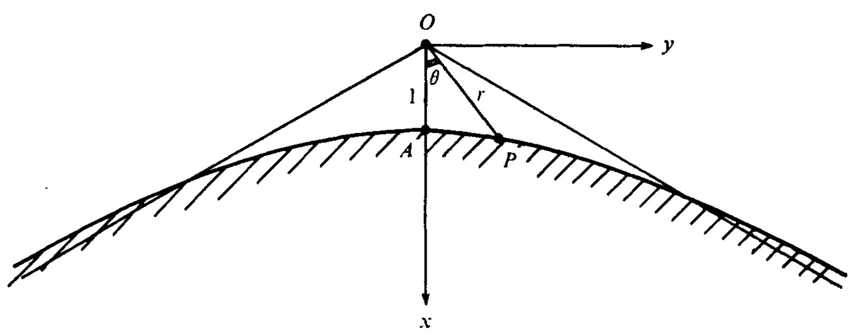

FIGURE 2. Axes and co-ordinates in the physical plane.

In Stokes's well-known corner flow (Stokes 1880) we have

$$i\chi = \tfrac{2}{3} g^{\frac{1}{2}} z^{\frac{3}{2}}, \tag{2.4}$$

and the free surface $\theta = \pm\frac{1}{3}\pi$ passes through the origin, which is a stagnation point. Now, on the other hand, we are seeking a solution in which the velocity q at the crest is different from zero. Thus we may choose units of length and time so that

$$g = 1, \quad q^2 = 2, \tag{2.5}$$

making the vertical distance of the origin O above the wave crest equal to unity also. Lastly we require that the flow shall tend to the Stokes corner flow at infinity, that is to say

$$i\chi \sim \tfrac{2}{3} z^{\frac{3}{2}} \quad \text{as} \quad r \to \infty. \tag{2.6}$$

3. A transformation of co-ordinates

Let us make the transformation

$$\tfrac{2}{3} z^{\frac{3}{2}} = \zeta = \rho\, e^{i\sigma}, \tag{3.1}$$

so that in effect we map the required flow onto the Stokes corner flow. In the ζ plane our required flow appears as in figure 3. From (2.1) and (3.1) we have

$$\rho = \tfrac{2}{3} r^{\frac{3}{2}}, \quad \sigma = \tfrac{3}{2}\theta, \tag{3.2}$$

so that the boundary condition (2.3) becomes

$$|d\chi/d\zeta|^2 = 2\cos(2\sigma/3), \tag{3.3}$$

with the condition

$$i\chi \sim \zeta \quad \text{as} \quad \zeta \to \infty. \tag{3.4}$$

This is obviously satisfied by the Stokes corner flow $i\chi = \zeta$ but only on the streamline $\sigma = \pm\frac{1}{2}\pi$ passing through the origin. In other words, no *interior* streamline of the Stokes corner flow is a line of constant pressure. Now in figure 3 we require that the free surface pass not through the origin but through the point

$$z = 1, \quad \zeta = \tfrac{2}{3} = \rho_0. \tag{3.5}$$

We note that in order for the free surface to be asymptotic to the line $\theta = \pm\frac{1}{3}\pi$ in figure 2 it is necessary only that $r|\theta - \frac{1}{3}\pi| \to 0$ as $r \to \infty$. In figure 3 this implies that $\rho^{\frac{2}{3}}|\sigma - \frac{1}{2}\pi| \to 0$ or that

$$\rho|\sigma - \tfrac{1}{2}\pi| = o(\rho^{\frac{1}{3}}). \tag{3.6}$$

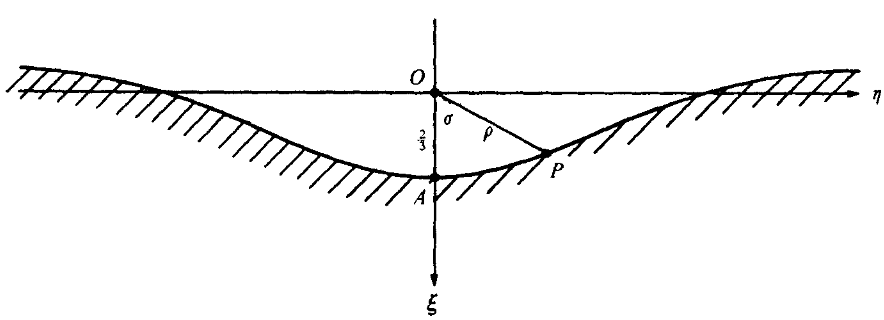

FIGURE 3. Axes and co-ordinates in the plane of $\zeta = \frac{2}{3} z^{\frac{3}{2}}$.

In other words it is still permissible for the free-surface streamline in figure 3 to diverge from the line $\sigma = \frac{1}{2}\pi$ by an amount that is $o(\rho^{\frac{1}{3}})$ as $\rho \to \infty$. If on the other hand we impose the stronger condition that

$$\rho|\sigma - \tfrac{1}{2}\pi| = O(1),$$

that is, the displacement of the surface streamline in the ζ plane is *bounded*, then it follows that

$$r|\theta - \tfrac{1}{3}\pi| = O(r^{-\frac{1}{2}}), \tag{3.7}$$

in other words, in the physical plane, the displacement of the free surface will tend to zero like $r^{-\frac{1}{2}}$.

4. Asymptotic behaviour at infinity

Let us now examine more closely the behaviour of the flow as $r \to \infty$. Suppose first that

$$i\chi \sim \zeta + iP - Q/\zeta^{\lambda}, \tag{4.1}$$

where P, Q and λ are real constants, with $\lambda > 0$. The first two terms on the right represent a uniform flow, and the third term a small perturbation of order $\rho^{-\lambda}$ as $\rho \to \infty$. From the real part of (4.1),

$$-\psi \sim \rho \cos\sigma - Q\rho^{-\lambda} \cos\lambda\sigma. \tag{4.2}$$

Hence on the free surface $\psi = 0$ the normal displacement $\xi \equiv \rho\cos\sigma$ is given by

$$\xi \sim Q\rho^{-\lambda}\cos\lambda\sigma. \tag{4.3}$$

So

$$\bar{\sigma} \equiv \tfrac{1}{2}\pi - \sigma \sim \xi/\rho \sim Q\rho^{-(\lambda+1)}\cos\lambda\sigma.$$

From (4.2) we calculate, with neglect of $\rho^{-2(\lambda+1)}$,

$$|d\chi/d\zeta|^2 = (\partial\psi/\partial\rho)^2 + (\rho^{-1}\partial\psi/\rho\,\partial\sigma)^2 \sim 1 + 2\lambda Q\rho^{-(\lambda+1)}\cos(\lambda+1)\sigma,$$

and

$$2\cos(2\sigma/3) \sim 1 + \sqrt{3}\sin(2\bar{\sigma}/3) \sim 1 + (2/\sqrt{3})\,\xi/\rho \sim 1 + (2/\sqrt{3})\,Q\rho^{-(\lambda+1)}\cos\tfrac{1}{2}\lambda\pi$$

when $\psi = 0$. Thus the boundary condition (3.3) is satisfied to order $\rho^{-(\lambda+1)}$ provided that

$$\lambda\cos\tfrac{1}{2}(\lambda+1)\pi = (1/\sqrt{3})\cos\tfrac{1}{2}\lambda\pi,$$

that is

$$-\lambda\sin\tfrac{1}{2}\lambda\pi = (1/\sqrt{3})\cos\tfrac{1}{2}\lambda\pi,$$

or

$$\tfrac{1}{2}\lambda\pi\tan\tfrac{1}{2}\lambda\pi = -\pi/2\sqrt{3}. \tag{4.4}$$

The smallest positive root of this equation is

$$\tfrac{1}{2}\lambda\pi = 2{\cdot}8316, \tag{4.5}$$

corresponding to

$$\lambda = 1{\cdot}8027. \tag{4.6}$$

It is interesting to note that in examining the form of the *highest wave*, in the neighbourhood of the wave crest, Grant (1973) arrived at an expansion of the form

$$z \sim A(i\chi)^{\frac{2}{3}} + B(i\chi)^{\nu},$$

where A, B and ν were constants with ν satisfying

$$\tan \tfrac{1}{2}\nu\pi = -(4+3\nu)/3\surd 3\,\nu. \tag{4.7}$$

At first sight (4.7) appears more complicated than (4.4). However, on writing

$$\nu = -(\lambda + \tfrac{1}{3}) \tag{4.8}$$

the reader will find, after some working, that (4.7) reduces to (4.4) precisely. Grant (1973) was interested only in expansions for z valid in the neighbourhood of the crest ($z \to 0$). He therefore calculated the smallest root of (4.7) greater than $\frac{2}{3}$. By (4.8), this corresponds to the smallest root of (4.4) less than -1, namely $\lambda = -1{\cdot}8027$. Other roots of (4.7) appropriate to Grant's problem correspond to negative roots of (4.4).

Equation (4.4) evidently has an infinity of real positive roots, each corresponding to a surface perturbation vanishing algebraically as $\rho \to \infty$. In addition there are two imaginary roots $\lambda = \pm i\mu$, say. To find the significance of these, let us assume, instead of (4.1), that

$$i\chi \sim \zeta + iP - Q/2\zeta^{i\mu} - Q^*/2\zeta^{-i\mu}, \tag{4.9}$$

where P and μ are real, $Q = A\,e^{i\epsilon}$ and $Q^* = A\,e^{-i\epsilon}$. The flow is still symmetric about the line $\sigma = 0$ and we now have

$$-\psi \sim \rho\cos\sigma - A\cosh\mu\sigma\cos(\mu\ln\rho - \epsilon). \tag{4.10}$$

Therefore on the free surface

$$\xi \sim B\cos(\mu\ln\rho - \epsilon), \tag{4.11}$$

where B is written for $A\cosh\frac{1}{2}\mu\pi$. On applying the free-surface condition (3.3) as before we now obtain

$$\tfrac{1}{2}\mu\pi\tanh\tfrac{1}{2}\mu\pi = \pi/2\surd 3. \tag{4.12}$$

This would also result from writing $\lambda = i\mu$ in (4.4). The only positive root of (4.12) is

$$\tfrac{1}{2}\mu\pi = 1{\cdot}1220, \tag{4.13}$$

giving

$$\mu = 0{\cdot}7143. \tag{4.14}$$

In this solution the perturbation ξ is oscillatory by (4.11), but it is bounded, so that in the physical plane the surface displacement is $O(r^{-\frac{1}{2}})$ at infinity. This mode evidently dominates over the modes corresponding to real positive roots of (4.4), since these vanish like $r^{-(3\lambda+1)/2}$, which for the smallest root (4.6) is $r^{-3{\cdot}204}$.

The most general asymptotic expression of the form (4.1) or (4.9) satisfying the condition of symmetry about the line $\sigma = 0$ is

$$i\chi \sim \zeta + iP - Q/\zeta^{\lambda} - Q^*/\zeta^{\lambda^*}, \tag{4.15}$$

where λ is complex, and λ^* is its conjugate. But it is easy to show that there exists no physically acceptable solution of the form (4.15), satisfying the boundary condition (3.3), apart from those already found.

To summarize, the asymptotic behaviour of χ as $r \to \infty$ is given by (4.9), provided that μ satisfies the characteristic equation (4.12). The only positive root of (4.12) corresponds to an oscillation which decays at large distances like $r^{-\frac{1}{2}}$. The negative imaginary roots of (4.12) correspond to perturbations which decay more rapidly than $r^{-\frac{1}{2}}$. The positive imaginary roots of (4.12) correspond to perturbations which tend to ∞ with r but tend to 0 as $r \to 0$. They are relevant to Grant's problem, namely the expansion of the *highest* wave in the neighbourhood of the sharp corner.

Equation (4.12) bears an obvious resemblance to the dispersion relation

$$(2\pi h/L)\tanh(2\pi h/L) = \omega^2 h/g$$

which occurs in the Stokes theory of infinitesimal waves of length L and radian frequency ω in water of mean depth h.

5. An approximation by dipoles

To obtain an approximation valid over the central range of ϕ we return to figure 3 and note that the flow in the lower half-plane may be roughly represented by a uniform flow together with a dipole situated at some point directly above the wave crest:

$$i\chi = \zeta - A/(\zeta + d). \tag{5.1}$$

Here A and d are real constants to be chosen so as to satisfy the boundary conditions at some point on the surface, say the crest $\zeta = \rho_0 = \frac{2}{3}$. At this point we have

$$\psi = 0 \quad \text{and} \quad |d\chi/d\zeta|^2 = 2. \tag{5.2}$$

Substitution from (5.1) yields respectively

$$A/(d+\rho_0) = \rho_0 \quad \text{and} \quad 1 + A/(d+\rho_0)^2 = \sqrt{2}.$$

So

$$d = \sqrt{2}\,\rho_0, \quad A = (\sqrt{2}+1)\rho_0^2. \tag{5.3}$$

The equation of the free surface $\psi = 0$ is then

$$\xi = A(\xi+d)/[(\xi+d)^2 + \eta^2], \tag{5.4}$$

where $\xi + i\eta = \zeta$. This represents a cubic curve in the ξ, η plane.

Closer approximations may be obtained by placing a sequence of dipoles along the negative ξ axis, so representing the cut in the ζ plane. The positions of the dipoles may be chosen so as to be regularly and densely distributed over the negative ξ axis. We may also adjust the position of the origin by writing

$$z' = z + D, \quad \zeta' \equiv \rho' e^{i\sigma'} = \tfrac{2}{3} z'^{\frac{3}{2}}. \tag{5.5}$$

Hence we set

$$i\chi = \zeta' - \sum_{m=1}^{M} \frac{A_m}{\zeta' + d_m}, \tag{5.6}$$

where

$$d_m = \rho_0' \tan\{m\pi/2(M+1)\}, \qquad m = 1, 2 \ldots M, \tag{5.7}$$

and the constants $A_1 \ldots A_M$ and D are to be chosen to satisfy the Bernoulli condition

$$|d\chi/d\zeta'|^2 = 2\cos(2\sigma'/3) - 2D(3\rho'/2)^{-\frac{2}{3}} \tag{5.8}$$

at M suitably chosen points on the free surface $\psi = 0$, say when $\sigma' = j\pi/2M$ ($j = 0$, $1, \ldots, M-1$), together with the scaling condition

$$\rho' = \rho_0' = \tfrac{2}{3}(1+D)^{\frac{3}{2}} \quad \text{when} \quad \sigma' = 0. \tag{5.9}$$

Then by rescaling with respect to ρ_0' we obtain a system of equations which is easily solved numerically. The resulting profiles are found to converge rapidly (see figure 4). The values of $A_1, \ldots, A_M$ and $d_1, \ldots, d_M$ when $M = 7$ are given in table 1. We have also

$$D = 0{\cdot}21225, \quad \rho_0' = 0{\cdot}88981.$$

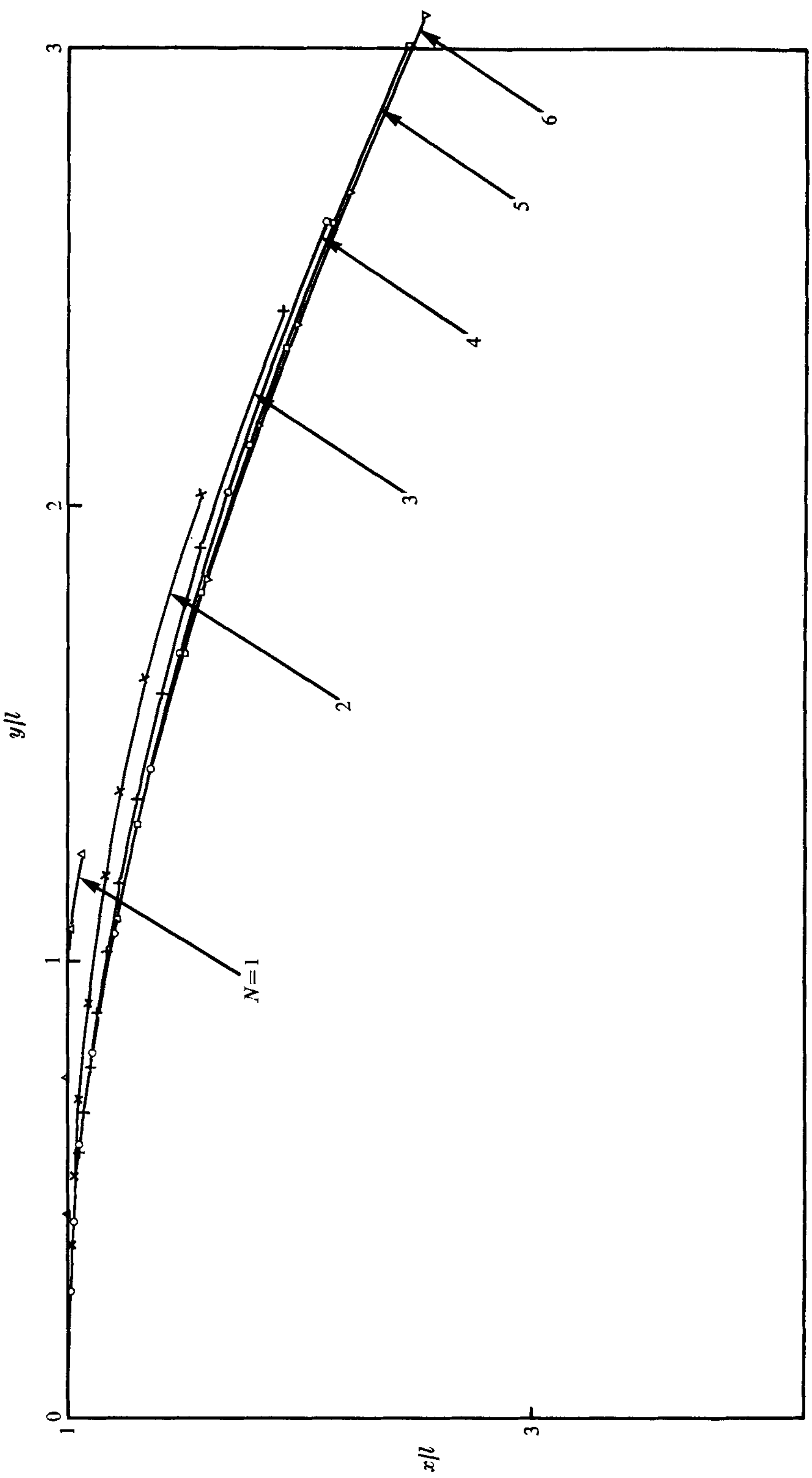

FIGURE 4. Successive approximations to the free surface by the dipole series (5.6).

m	d_m/ρ_0'	$A_m/\rho_0'^2$
1	0·19891	0·06591
2	0·41421	−0·15294
3	0·66818	0·61481
4	1·00000	0·76622
5	1·49661	1·22932
6	2·41421	−0·18775
7	5·02734	3·79920

TABLE 1. Positions and strengths of the dipoles in the flow given by (5.7), when $M = 7$.

For any fixed value of M the dipole terms in (5.6) behave like ζ^{-1} for large ζ, so the approximation does not have the correct behaviour at infinity. Nevertheless we shall see that for finite values of M the resulting profile provides an excellent check on the independent method of calculation to be described in §6.

6. Calculation of the complete flow

We shall now calculate the complete solution by a method similar to that used by Michell (1893) for the profile of the highest progressive wave in deep water, and by Lenau (1966) for the highest solitary wave. Let us take as co-ordinates the potential ϕ and stream function ψ (in the steady motion) and attempt to calculate the complex variable $z = re^{i\theta}$ as an analytic function of $\chi = \phi + i\psi$. z must be regular throughout the half-plane $\psi < 0$ and at all points on the boundary. Also z must be symmetric about the line $\phi = 0$. The free-surface condition (2.3) can be written

$$\mathrm{Re}\{z|dz/d\chi|^2\} = \tfrac{1}{2}. \tag{6.1}$$

When $\chi \to \infty$ in the lower half-plane the solution must approach the Stokes corner flow. Hence

$$z \sim (\tfrac{3}{2}i\chi)^{\frac{2}{3}} \quad \text{as} \quad \chi \to \infty,$$

or equivalently

$$z/(\delta + i\chi)^{\frac{2}{3}} \to (\tfrac{3}{2})^{\frac{2}{3}} \quad \text{as} \quad \chi \to \infty, \tag{6.2}$$

where δ is any fixed positive constant. We also specify that $z/(\delta+i\chi)^{\frac{2}{3}}$ shall be of bounded variation on the surface $\psi = 0$.

We now transform the lower half-plane of χ onto the interior of the unit circle in the plane of a new variable ω (see figure 5) by writing

$$i\chi = \beta(1-\omega)/(1+\omega), \qquad \omega = (\beta - i\chi)/(\beta + i\chi), \tag{6.3}$$

where β is some real, positive constant. The wave crest A and the point at infinity in the χ plane correspond to the points $\omega = 1$ and $\omega = -1$ respectively. The centre C of the circle in the ω plane corresponds to the point $\chi = -i\beta$ on the negative ψ axis. The point $\chi = i\beta$ corresponds to the point E at infinity in the ω plane.

Now $z/(\delta+i\chi)^{\frac{2}{3}}$ is analytic inside and on the circle $|\omega| = 1$, except at $\omega = -1$, and so has an expansion in powers of ω. Thus

$$z = (\delta + i\chi)^{\frac{2}{3}}(b_0 + b_1\omega + b_2\omega^2 + \ldots), \tag{6.4}$$

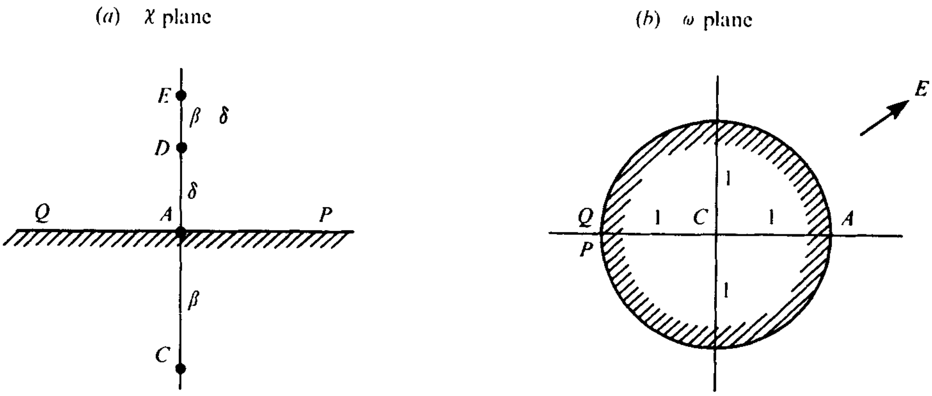

FIGURE 5. Representation of the flow (a) in the plane of $\chi = \phi + i\psi$ and (b) in the ω plane.

where the b_n are real by symmetry. From the assumption of bounded variation the series converges absolutely and uniformly in and on the circle $|\omega| = 1$. Now

$$\delta + i\chi = \beta[\alpha + (1-\omega)/(1+\omega)], \tag{6.5}$$

where $\alpha = \delta/\beta$. So (6.4) can be written

$$z = \beta^{\frac{2}{3}}[\alpha + (1-\omega)/(1+\omega)]^{\frac{2}{3}}(b_0 + b_1\omega + b_2\omega^2 + \ldots) \tag{6.6}$$

and we may specify that the argument of the radical lies between $\pm\frac{1}{3}\pi$.

Formally differentiating each side of (6.4) we have

$$dz/d\chi = \tfrac{2}{3}i(\delta+i\chi)^{-\frac{1}{3}}(b_0 + b_1\omega + b_2\omega^2 + \ldots) \\ + (\delta + i\chi)^{\frac{2}{3}}(b_1 + 2b_2\omega + 3b_3\omega^2 + \ldots)\,d\omega/d\chi.$$

From (6.3)

$$d\omega/d\chi = (1+\omega)^2/2i\beta.$$

So

$$dz/d\chi = i(\delta+i\chi)^{-\frac{1}{3}}[\tfrac{2}{3}(b_0 + b_1\omega + b_2\omega^2 + \ldots) \\ - \tfrac{1}{2}\{\alpha + 1 + 2\alpha\omega + (\alpha-1)\,\omega^2\}(b_1 + 2b_2\omega + \ldots)]. \tag{6.7}$$

Provided that $(\delta + i\chi)^{\frac{1}{3}}dz/d\chi$ also is of bounded variation on $|\omega| = 1$, the above power series are absolutely convergent and square-integrable. So we may form the product $z|dz/d\chi|^2$ as a convergent Fourier series in $\tau = \arg\omega$, and, on substituting in (6.1), we have

$$\mathrm{Re}\left\{\left(\frac{\delta+i\chi}{\delta-i\chi^*}\right)^{\frac{1}{3}}(b_0 + b_1e^{i\tau} + b_2e^{2i\tau} + \ldots)[\tfrac{2}{3}(b_0 + b_1e^{i\tau} + b_2e^{2i\tau} + \ldots) \\ - \tfrac{1}{2}(b_1 + 2b_2e^{i\tau} + 3b_3e^{2i\tau} + \ldots)\{\alpha + 1 + 2\alpha e^{i\tau} + (\alpha-1)\,e^{2i\tau}\}][\ldots]^*\right\} = \tfrac{1}{2}. \tag{6.8}$$

Lastly, to expand the radical in a Fourier series on $|\omega| = 1$ we have, since

$$\omega^* = e^{-i\tau} = \omega^{-1},$$

$$\frac{\delta+i\chi}{\delta-i\chi^*} = \frac{\alpha + (1-\omega)/(1+\omega)}{\alpha + (\omega-1)/(\omega+1)} = \frac{(1+\alpha)-(1-\alpha)\,\omega}{(1+\alpha)\,\omega-(1-\alpha)}.$$

Hence

$$[(\delta+i\chi)/(\delta-i\chi^*)]^{\frac{1}{3}} = \omega^{-\frac{1}{3}}(1-\gamma\omega)^{\frac{1}{3}}(1-\gamma/\omega)^{-\frac{1}{3}}, \tag{6.9}$$

where

$$\gamma = (1-\alpha)/(1+\alpha). \tag{6.10}$$

The first factor on the right of (6.9) can be expanded as a Fourier series valid in $-\pi < \tau < \pi$:

$$e^{-\frac{1}{3}i\tau} = \sum_{n=-\infty}^{\infty} C_n e^{in\tau}, \quad C_n = \frac{\sin(n-\frac{1}{3})\pi}{(n-\frac{1}{3})\pi}, \tag{6.11}$$

and since $|\gamma| < 1$, both the second and third terms can always be expanded in power series uniformly convergent on the circle $|\omega| = 1$, namely

$$(1-\gamma\omega)^{\frac{1}{3}} = 1 - \tfrac{1}{3}\gamma e^{i\tau} - \tfrac{1}{9}\gamma^2 e^{2i\tau} - \dots \tag{6.12a}$$

$$(1-\gamma/\omega)^{-\frac{1}{3}} = 1 + \tfrac{1}{3}\gamma e^{-i\tau} + \tfrac{2}{9}\gamma^2 e^{-2i\tau} + \dots. \tag{6.12b}$$

Finally substituting these expressions into (6.8) and equating coefficients of $\cos n\tau$, where $n = 0, 1, 2, \dots$, we obtain a sequence of relations between the coefficients $b_0, b_1, b_2 \dots$. We impose also the scaling condition that $z = 1$ when $\chi = 0$. From (6.4) this gives

$$b_0 + b_1 + b_2 + \dots = \delta^{-\frac{2}{3}}. \tag{6.13}$$

This system of equations may be solved by truncation and successive approximation, assuming that $b_n = 0$ when $n > N$, say, where N is the order of the approximation. Then the coefficients of $1, \cos\tau, \dots, \cos(N-1)\tau$ in (6.9) together with the scaling condition (6.13) give us $N+1$ equations to determine $b_0, b_1, \dots, b_N$.

The choice of α and β, and hence $\gamma = (\alpha-1)/(\alpha+1)$ and $\delta = \alpha\beta$, are at our disposal. These may be selected so as to maximize the rate of convergence. However, experience showed that in fact the final solution was affected not at all, and the convergence only weakly, by the choice of α, so long as this was $O(1)$. From (6.10) it is obviously most convenient mathematically to take $\alpha = 1$ so $\gamma = 0$ and $\delta = \beta$. But in practice, with the aid of subroutines for handling power series, general values of α may be almost as easily accommodated. The value of β affects the relation (6.3) between χ and ω. A fixed point on the unit circle in the ω plane corresponds, for a large value of β, to a large value of χ, and for a small value of β to a small value of χ. Hence we expect that small values of β will give a more accurate representation of the profile near the wave crest, while large values of β will give a better representation of the 'tails' of the profile. Numerical solutions indicate that an optimum value of β, for a 40-term series, is around $10^{\frac{3}{2}}$.

7. Results of the calculation

The method of §6 was programmed in FORTRAN IV on the IBM 370-165 at Cambridge University, a standard subroutine being used to solve the nonlinear algebraic equations for $b_0, b_1, \dots, b_N$ (table 2). It was found that independently of the values

n	b_n	n	b_n	n	b_n
0	0·817769	7	−0·003474	14	−0·000365
1	−0·558142	8	−0·002572	15	−0·000206
2	−0·084588	9	−0·001566	16	−0·000211
3	−0·031527	10	−0·001257	17	−0·000109
4	−0·016685	11	−0·000764	18	−0·000126
5	−0·008899	12	−0·000660	19	−0·000058
6	−0·005894	13	−0·000391	20	−0·000078

TABLE 2. Coefficients b_n in the power series expansion (6.6), when $\alpha = 1$ and $\beta = 10 \cdot 0^{1 \cdot 5}$.

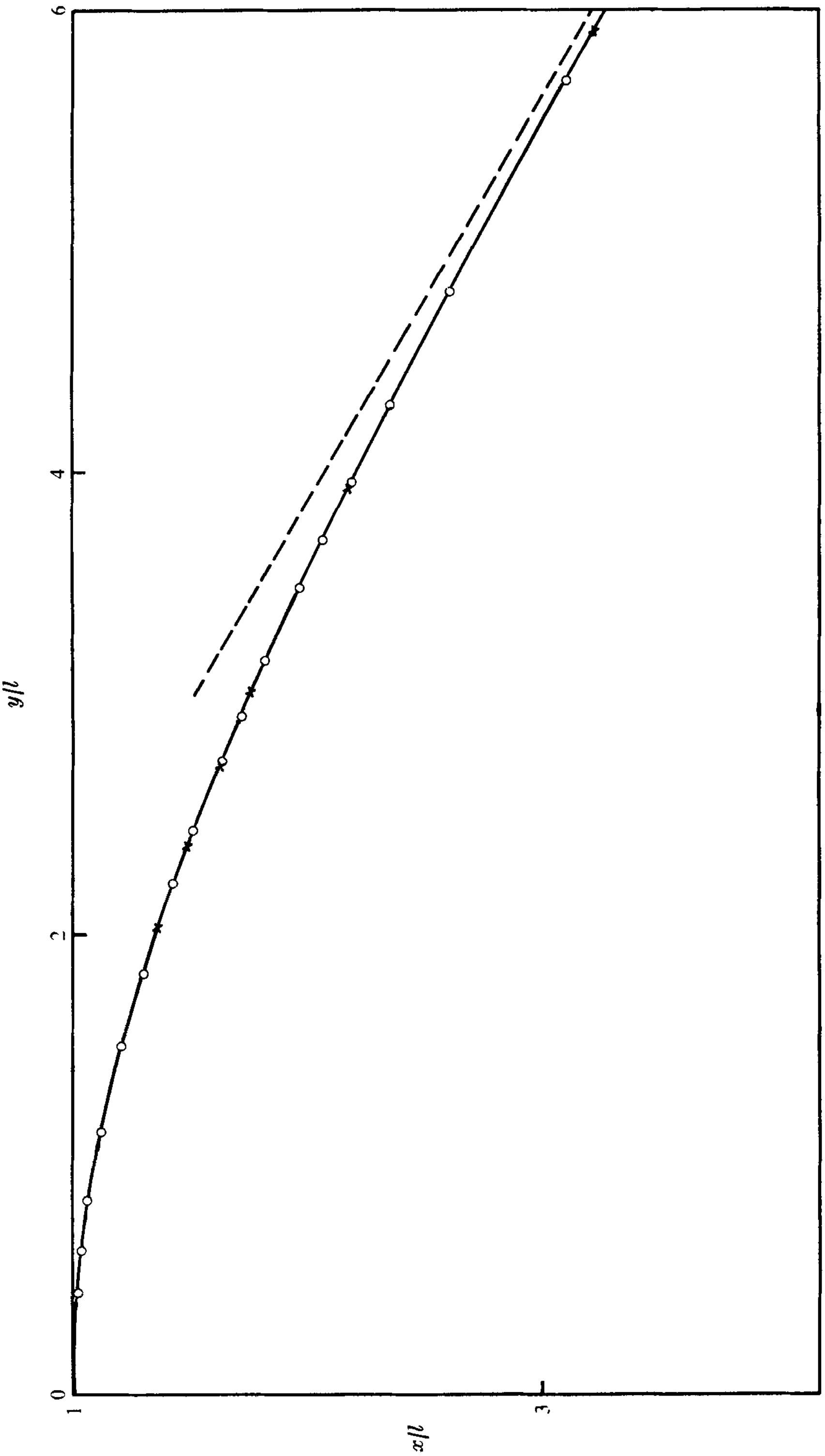

FIGURE 6. Comparison of the crest profile as found by two independent methods. ○, dipole approximation ($N = 7$); ×, Fourier series ($N = 60$). The broken line is the asymptote.

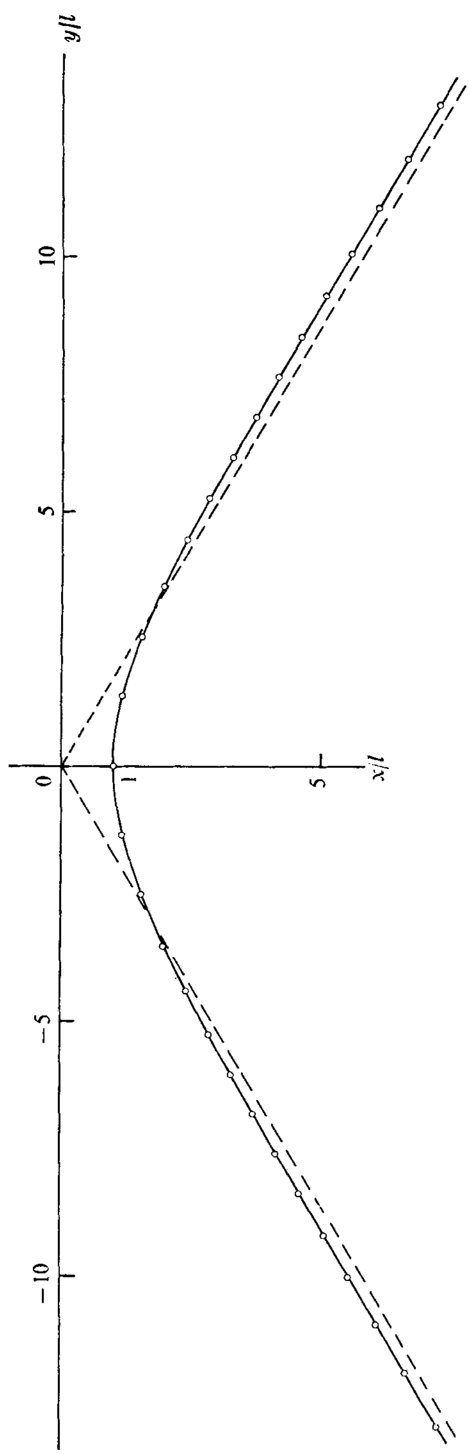

FIGURE 7. The profile of the free surface when $r/l \leqslant 15$, showing the first intersection with the asymptote. The angle at the origin is 120°.

of α or β the surface profile converged to a unique curve. The form near the wave crest is shown in figure 6. Also shown are the plotted points derived from the highest dipole approximation of § 5. It will be seen that the two sets of results are indistinguishable, thus providing a valuable check on the calculations and a strong indication that the solution to the problem is unique. Figure 7 shows the profile on a reduced scale. It appears at once that the surface crosses its asymptotes and then approaches them gradually from above. The asymptotic behaviour of the profile as $r \to \infty$ can be seen more clearly from figure 8, in which

$$\xi = \mathrm{Re}\{\zeta\} = \tfrac{2}{3}r^{\frac{3}{2}} \cos \tfrac{3}{2}\theta \tag{7.1}$$

has been plotted, for convenience, against $\ln r$. The particular values of the parameters are $\alpha = 0{\cdot}5$ and $\beta = 10{\cdot}0^{\frac{3}{2}}$. The curves corresponding to $N = 20$, 40 and 60 show that as

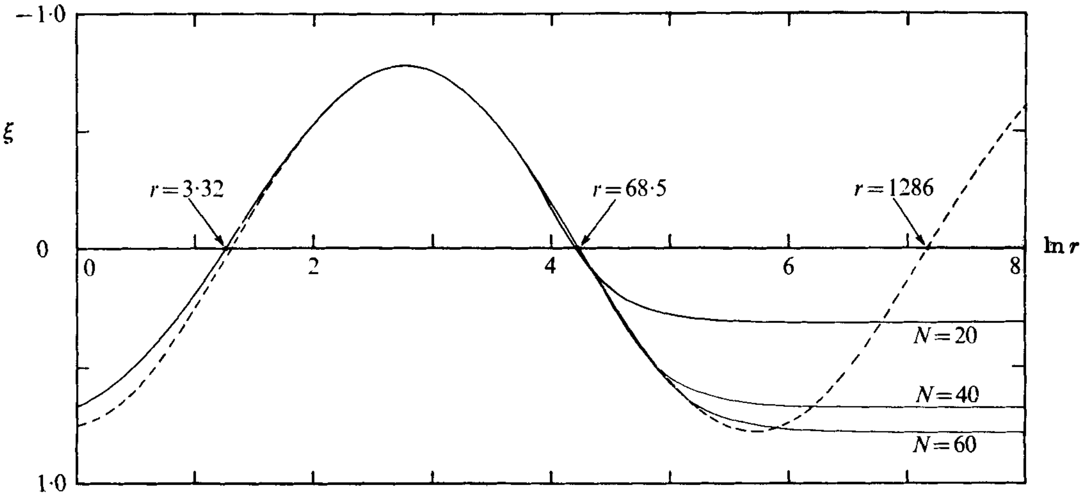

FIGURE 8. A plot of $\xi \equiv \mathrm{Re}\ \zeta$ against $\ln r$, for points on the free surface, when $N = 20$, 40 and 60. Parameters: $\alpha = 0{\cdot}5$ and $\beta = 10{\cdot}0$. The broken curve represents the sine wave (4.11) with $B = 0{\cdot}78$ and $\epsilon = -10{\cdot}3°$.

N increases the approximations are tending towards a limiting curve, which crosses the asymptote again at $\ln r = 4{\cdot}23$, or $r = 68{\cdot}5$. From (4.11) one would expect that asymptotically ξ would oscillate harmonically in $\ln r$ with wavelength

$$2\pi/\tfrac{3}{2}\mu = 4\pi/3\mu = 5{\cdot}864, \tag{7.2}$$

since $\mu = 0{\cdot}7143$. In figure 8 the dashed curve represents a pure sine wave of exactly this wavelength which has been adjusted in amplitude and phase so as to pass through the calculated crossing at $\ln r = 4{\cdot}23$ and through the maximum at about $\ln r = 2{\cdot}8$. The appropriate values of the constants in (4.11) are

$$B = 0{\cdot}78, \quad \epsilon = -0{\cdot}180\,\mathrm{rad.} = -10{\cdot}3°. \tag{7.3}$$

From this analysis it may be inferred that there are further crossings of the asymptote at regularly spaced values of $\ln r$, the next being at $\ln r = 7{\cdot}16$ or $r = 1286$.

Figure 9 shows the surface profile in the physical plane on a very small scale so that the second crossing of the asymptote, at $r = 68{\cdot}5$, can just be discerned. At still greater values of r the profile is not graphically distinguishable from its asymptote.

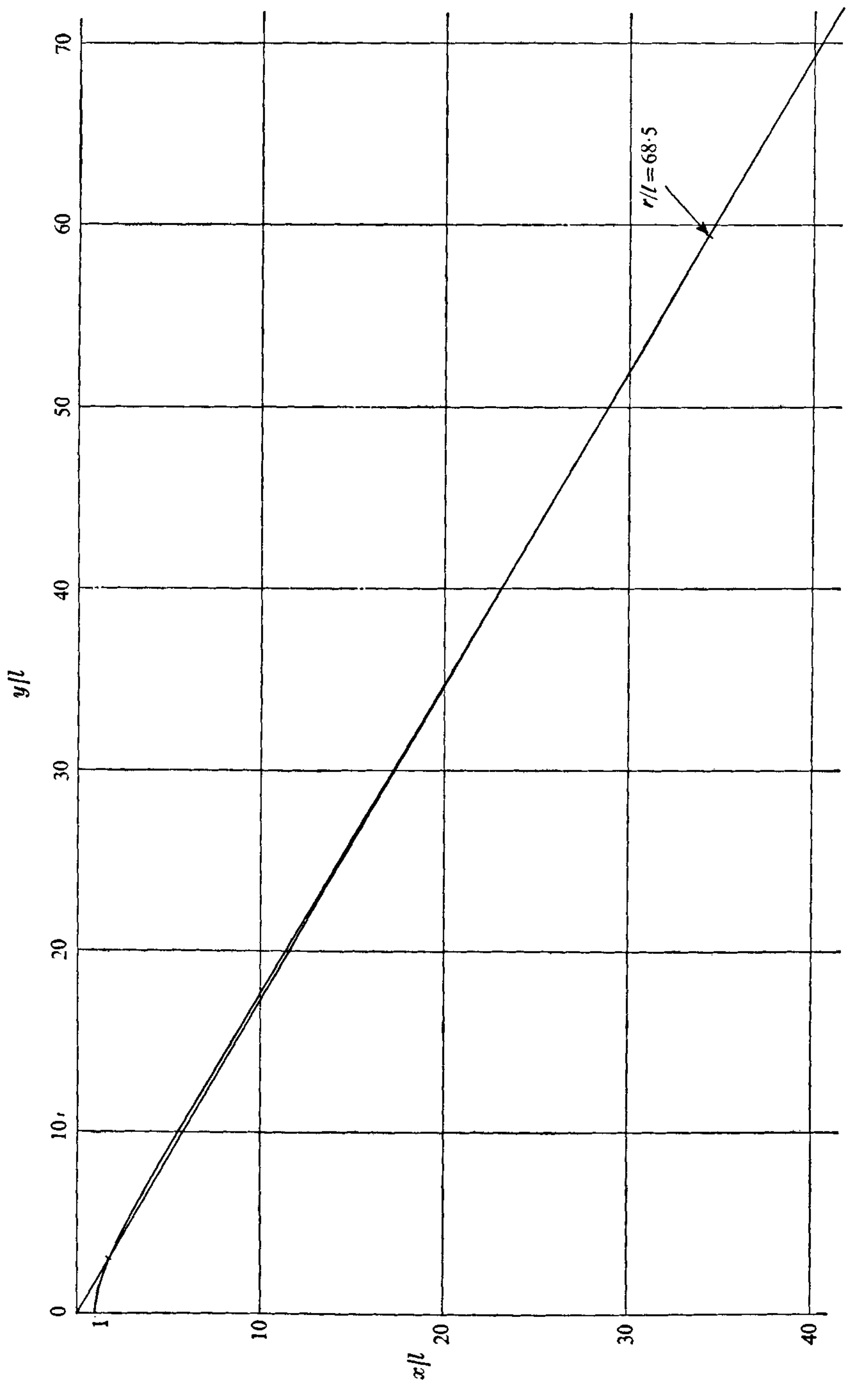

FIGURE 9. The profile of the free surface when $r/l \leqslant 75$. The second crossing of the asymptote occurs at $r/l = 68{\cdot}5$.

Since $\xi = \frac{2}{3}r^{\frac{3}{2}} \cos \frac{3}{2}\theta$, an asymptotic expression for the free-surface profile is given by

$$\cos \tfrac{3}{2}\theta = \tfrac{3}{2}Br^{-\frac{3}{2}} \cos (\tfrac{3}{2}\mu \ln r - \epsilon), \tag{7.4}$$

where B and ϵ are given by (7.3). From figure 8 we see that this expression is a good approximation not only for large values of r but over the whole range $1 \leqslant r < \infty$.

8. The maximum slope

In some analytical studies of symmetric water waves (Krasovskii 1961; Keady & Pritchard 1974) it has been assumed that the maximum slope angle of the free surface does not exceed the value 30° corresponding to the Stokes corner flow. Thus it is interesting to note from figure 9 that, in the region where the free surface lies outside the asymptote, the maximum slope slightly exceeds 30°. The precise value is 30·37°. Some confirmation is provided by the recent calculations of Sasaki & Murakami (1973) on steep solitary waves and periodic waves in deep water. In figure 10 we have plotted

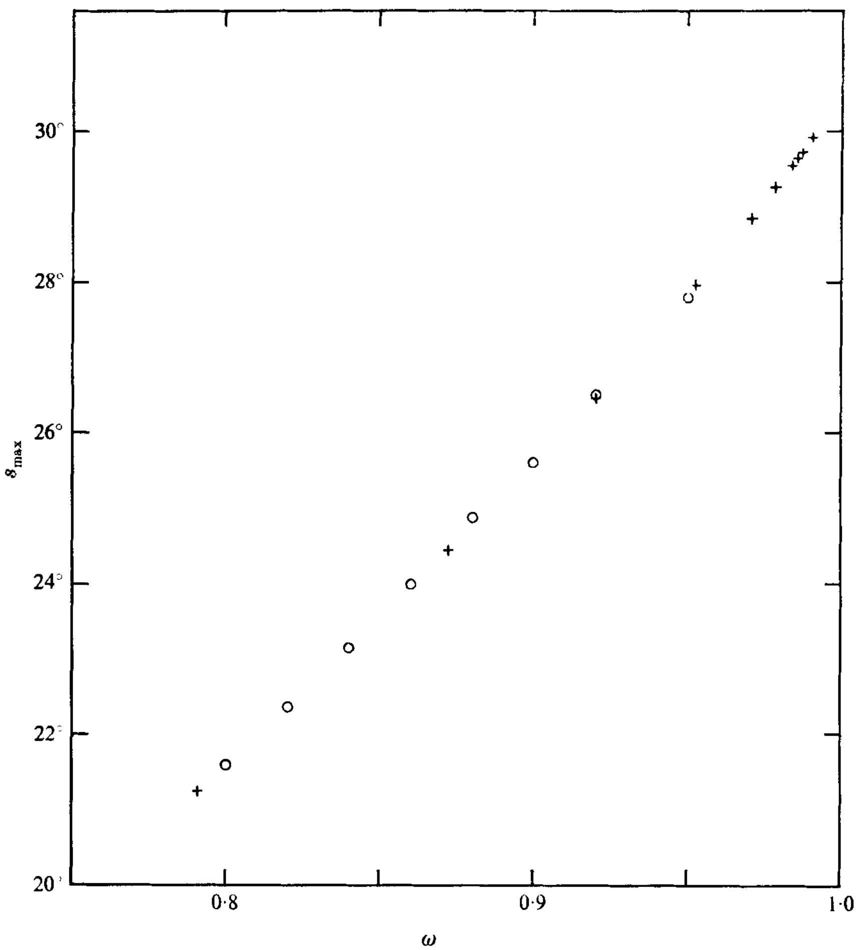

FIGURE 10. Comparison of the maximum slope of solitary waves as a function of ω. +, Sasaki & Murakami (1973); ○, Byatt-Smith & Longuet-Higgins (1976).

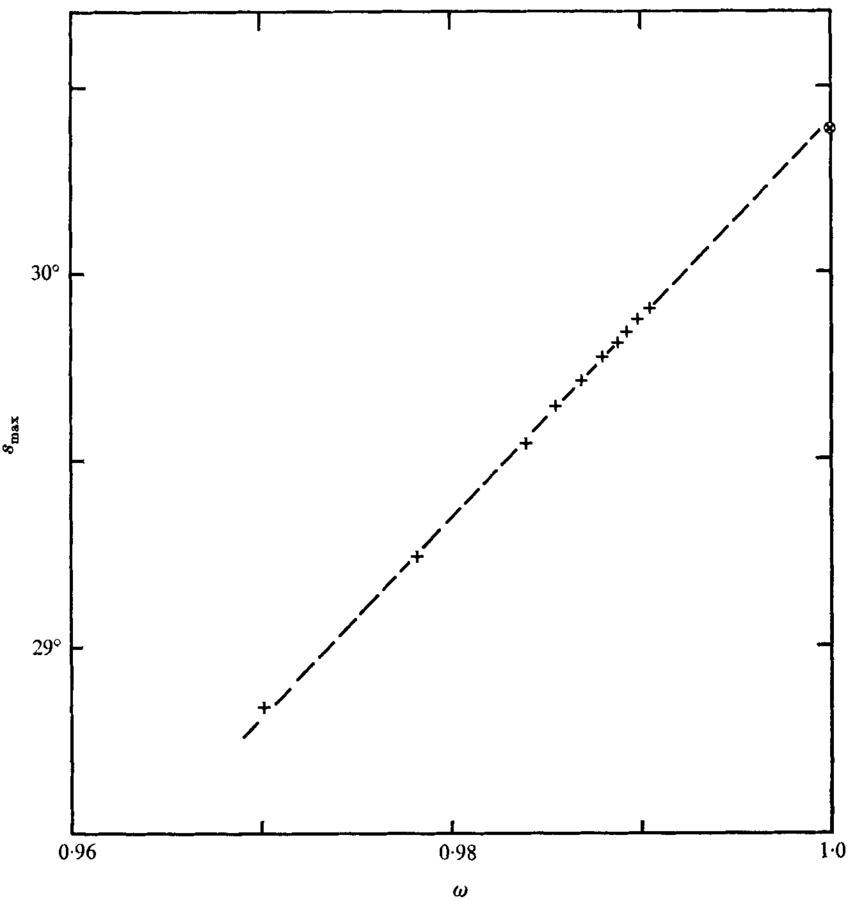

FIGURE 11. The maximum slope of solitary waves, as a function of ω (scale enlarged). +, from calculations of Sasaki & Murakami (1973); ⊗, from asymptotic profile of figure 9.

their values for the maximum slope s_{max} in solitary waves against the parameter ω defined in § 1. On the same graph are plotted some values calculated by Byatt-Smith & Longuet-Higgins (1976), showing that the two sets of calculations are consistent. Though none of the plotted values actually exceeds 30°, a linear extrapolation from the values of Sasaki & Murakami (see figure 11) intersects the limiting axis $\omega = 1$ very close to the value that we have just found.

A similar comparison can also be made for progressive waves in deep water. In figure 12 we have plotted the results of Sasaki & Murakami for this case against the parameter

$$\omega' = 1 - q^2 q'^2 / c^2 c_0^2,$$

introduced by Longuet-Higgins (1975). Here q and q' denote the particle speeds at the crest and trough respectively, in the steady flow relative to a frame moving with the wave speed c, and c_0 denotes the speed of infinitesimal waves having the same wavelength. A linear extrapolation from the plotted points in figure 11 passes even closer to the point on the axis $\omega = 1$ corresponding to our value of s_{max}. This confirms that the solution we have found does indeed represent a locally valid asymptotic form for steep gravity waves, whether in deep or in shallow water.

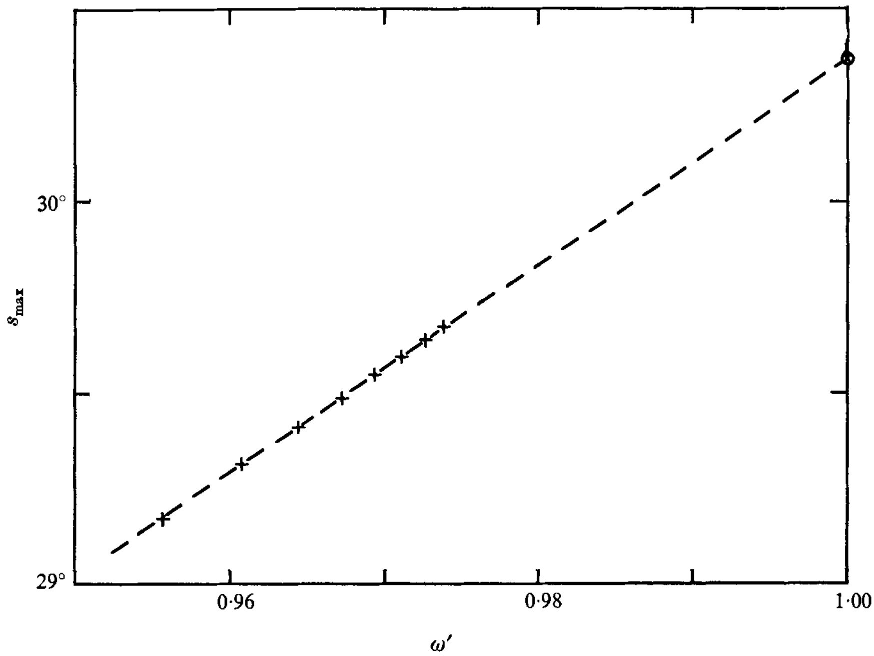

FIGURE 12. The maximum slope of deep-water waves, as a function of ω'. +, from calculations of Sasaki & Murakami (1973); ⊗, from asymptotic profile of figure 9.

9. Acceleration at the crest

In Stokes's corner flow the acceleration of the fluid at the free surface is the same as that of a particle sliding freely down an inclined plane at an angle of 30° to the horizontal. Thus the acceleration equals $\frac{1}{2}g$, directed down the slope. In the interior of the fluid it can be shown (see Longuet-Higgins 1963) that the magnitude of the acceleration is $\frac{1}{2}g$ everywhere, with the direction always radially away from the crest. So the acceleration vector has a strong singularity at the sharp corner.

In the present solution, the velocity and acceleration are everywhere smooth. The acceleration at the crest is given by

$$W = q^2/R, \tag{9.1}$$

where q is the particle velocity at the crest and R is the radius of curvature of the profile. In our scaling $g = 1$, $q^2 = 2$ and so

$$W = 2gl/R. \tag{9.2}$$

From the series of §6, summed with the help of Padé approximants to 40 terms, we find $R = 5{\cdot}15$ and hence

$$W = 0{\cdot}388g \tag{9.3}$$

directed vertically downwards. This is to be compared with the values found by Sasaki & Murakami for their steepest solitary and progressive waves, namely $0{\cdot}379g$. In the far field, as $r/l \to \infty$, the acceleration tends to the value $\frac{1}{2}g$ appropriate to the Stokes corner flow.

τ/π	x/l	y/l	τ/π	x/l	y/l
0·00	1·0000	0·0000	0·50	6·3508	11·4223
0·02	1·0464	0·6953	0·52	6·6378	11·9140
0·04	1·1709	1·3539	0·54	6·9383	12·4282
0·06	1·3448	1·9607	0·56	7·2541	12·9681
0·08	1·5445	2·5182	0·58	7·5876	13·5376
0·10	1·7564	3·0347	0·60	7·9412	14·1414
0·12	1·9732	3·5187	0·62	8·3181	14·7847
0·14	2·1915	3·9772	0·64	8·7221	15·4740
0·16	2·4097	4·4160	0·66	9·1577	16·2170
0·18	2·6271	4·8393	0·68	9·6302	17·0232
0·20	2·8438	5·2508	0·70	10·1467	17·9043
0·22	3·0598	5·6534	0·72	10·7155	18·8750
0·24	3·2757	6·0495	0·74	11·3476	19·9541
0·26	3·4919	6·4413	0·76	12·0572	21·1660
0·28	3·7090	6·8307	0·78	12·8632	22·5432
0·30	3·9274	7·2193	0·80	13·7911	24·1299
0·32	4·1480	7·6090	0·82	14·8769	25·9879
0·34	4·3712	8·0011	0·84	16·1724	28·2069
0·36	4·5978	8·3972	0·86	17·7561	30·9224
0·38	4·8286	8·7989	0·88	19·7536	34·3511
0·40	5·0643	9·2078	0·90	22·379	38·864
0·42	5·3059	9·6255	0·92	26·038	45·161
0·44	5·5542	10·0539	0·94	31·606	54·759
0·46	5·8103	10·4947	0·96	41·47	71·79
0·48	6·0754	10·9500	0·98	65·85	113·97

TABLE 3. Cartesian co-ordinates of the free surface.

10. Conclusion

The Cartesian co-ordinates of the surface profile are given in table 3. Because the length scale l is independent of g, we have effectively found a family of self-similar flows, each tending to the Stokes corner flow at infinity. At any fixed position in the physical plane, when $l \to 0$, the flow tends also to the Stokes flow. We have found empirically (see figure 1) that the surface profile agrees with that found from a calculation of a complete solitary wave. Further corroboration, both for solitary waves and progressive waves in deep water, comes from the maximum angle of slope (figures 11 and 12). The fact that the maximum slope very slightly exceeds 30° will necessitate the reconsideration of some earlier proofs of the existence of progressive gravity waves of finite amplitude.

It remains to be shown how the present solution can be used as an 'inner' solution, valid near the crest and matched asymptotically to an outer solution representing the remainder of the wave, so providing an independent method of calculation for steep gravity waves. This will be done in a paper to follow.

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